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On the multiplicity of periodic solutions for semilinear parabolic equations

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Abstract.
In the present paper, we consider the multiple existence of T-periodic solutions of semilinear parabolic equations.

1. Introduction.

Let $\Omega \subset R^N$ be a bounded domain with a smooth boundary $\partial\Omega$. Let L be a second order uniformly strongly elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j})$$

where the cofficient functions $a_{ij} = a_{ji}$ are real valued functions in $L^\infty(\Omega)$ and satisfies

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq C \mid \xi \mid^2 \qquad \text{for all } \xi \in R^n \text{ and } x \in \Omega$$

for some $C > 0$. We impose the Dirichlet boundary condition on L. That is

$$D(L) = \{u \in L^2(\Omega) : Lu \in L^2(\Omega), \qquad u(x) = 0 \qquad \text{on } \partial\Omega\}$$

Our purpose in note is to report on the multiple existence result of solutions for the problem of the form

$$(P) \qquad \begin{aligned} &\frac{du}{dt} + Lu - g(u) = f(t), \quad t > 0 \\ &u(0) = u(T), \end{aligned}$$

Here $T > 0$, $f : [0, \infty) \to L^2(\Omega)$ is a T-periodic function and $g : R \to R$ is a continuous function with $g(0) = 0$.

The existence of periodic solutions for problems of this kind has been studied by many authors.(See Amann[1] which also contains many references.) For the multiple existence of the periodic solutions, Amann[1] established a multiplicity result for the problem (P). To find a solution of (P), we can make use of two approaches. One way is to work with Poincare map and find fixed points. Another way is to find sub- and supersolutions of the problem (P). If one can find a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (P) satisfying $\underline{u} < \overline{u}$, there exists a solution of (P) between $\underline{u}$ and $\overline{u}$. The method employed in [1] is based on the super-subsolution method. In [6], the author considered the multiple exitstence of solutions of (P) by usig the Schauder's fixed point theorem and results for multiple solutions of nonlinear elliptic equations(cf. [2], [3], and [4]). In the present paper, we study the multiplicity of solutions for (P) by using the argument in [6] and the degree theory for compact mappings.

To state our result, we need some notations. We denote by $|\cdot|$ the norm of $L^2(\Omega)$. $0 < \lambda_1 < \lambda_2 \leq \cdots$ stand for the eigenvalues of the self-adjoint realization in $L^2(\Omega)$ of L. The norm of $H_0^1(\Omega)$ is given by

$$\| v \|^2 = \langle Lv, v \rangle \qquad \text{for } v \in H_0^1(\Omega).$$

The norm defined above is an equivalent norm with the usual norm of $H_0^1(\Omega)$. $W^{1,p}(0,T;X)(1 \leq p \leq \infty)$ stands for the space of functions $u \in L^p(0,T;X)$ with $du/dt \in L^p(0,T;X)$, where du/dt is the derivative in the sence of distribution.

We can now state our main result.

Theorem . *Suppose that g satisfies the following conditions:*

$$\text{(g1)} \qquad \sup_{t \in R} g'(t) < \lambda_2,$$

(g2) $$g'(\pm\infty) < \lambda_1 < g'(0) < \lambda_2,$$

where $g'(\pm\infty) = \lim_{t\to\pm\infty} g'(t)$. *Then there exists* $M > 0$ *such that for each T-periodic function*

$$f \in W^{1,\infty}(0,T;L^2(\Omega)) \quad \textit{satisfying} \quad \sup\{|\ f(t)\ |: t \in [0,T]\} \le M,$$

problem (P) possesses at least three solutions in $W^{1,\infty}(0,T;L^2(\Omega))$.

Remark . For the existence of a perioidic solution of (P), we do not need (g2). In fact, the existence of periodic solution of (P) is known under much more weaker conditions than (g1).

2. Preliminaries.

In the following we assume that (g1) and (g2) hold. we set $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $V^* = H^{-1}(\Omega)$. We denote by $\langle\cdot,\cdot\rangle$ the pairing of V and V^*. $\|\cdot\|_*$ stands for the norm of $H^{-1}(\Omega)$. For each subset $A \subset V$, $int(A)$ denotes the set of interior point of A. For each $i \ge 1$, V_i denotes the subspace of $H_0^1(\Omega)$ spanned by the eigenfunctions corresponding to the eigenvalues $\{\lambda_1,\cdots,\lambda_i\}$, and φ_i is a normalized eigenfunction corresponding to λ_i. Then $\varphi_1 \in L^\infty(\Omega)$ and $V_1 = \{k\varphi_1 : k \in R\}$. P_i is the projection from H onto V_i for each $i \ge 1$.

We define a functional $F : V \to R$ by

$$F(v) = \frac{1}{2}\langle Lv, v\rangle - \int_\Omega \int_0^{v(x)} g(\tau)d\tau dx \qquad \text{for each } v \in V.$$

We set

$$A_c = \{v \in H_0^1(\Omega) : F(v) \le c\} \qquad \text{for each } c \in R.$$

Then the problem (P) can be rewritten as

$$u_t + F'(u) = f(t), \quad u(0) = u(T). \tag{2.1}$$

Lemma 1.
(1)
The set $\{s \in R : F(s\varphi_1) < 0\}$ *consists of at least two intervals :*

(2) *There exists $\omega > 0$ such that for each $w \in V_1$,*

$$< F'(v_1 + w) - F'(v_2 + w), v_1 - v_2 > \geq \omega \parallel v_1 - v_2 \parallel^2 \tag{2.2}$$

for all $v_1, v_2 \in V_1^\perp$.

Proof. Since $\lambda_1 < g'(0)$, we can see from the definition of F that if $| s |$ is sufficiently small, $F(s\varphi_1) < 0 (= F(0))$. This implies that the set $A_0 = \{s \in R : F(s\varphi_1) < 0\}$ is nonempty. It is easy to see from the continuity of F that D consists of open intervals. Then since $F(0) = 0$, the assertion (1) follows.

We put $\omega = 1 - g'(0)/\lambda_2$. Then since $\parallel v \parallel \geq \lambda_2 | v |$ for $v \in V_1^\perp$, we have that

$$\begin{aligned} < F'(v_1 + w) - F'(v_2 + w), v_1 - v_2 > &\geq \parallel v_1 - v_2 \parallel^2 - g'(0) \mid v_1 - v_2 \mid^2 \\ &\geq \omega \parallel v_1 - v_2 \parallel^2 \end{aligned}$$

for all $v_1, v_2 \in V_1^\perp$. ∎

Remark. The inequality (2.2) implies that for each $w \in V_1$, the functional $F(\cdot + w) : V_1^\perp \to R$ is strictly convex.

Let u_- and u_+ be elements of $H_0^1(\Omega)$ such that

$$F(u_-) = \min\{F(v) : v \in V, < P_i v, \varphi_1 >< 0\},$$

and

$$F(u_+) = \min\{F(v) : v \in V, < P_i v, \varphi_1 >> 0\}.$$

From Lemma 1, u_- and u_+ are well defined and there exist open intervals (a_-, b_-) and (a_+, b_+) such that

$$P_1 u_- \in \{c\varphi_1 : a_- < c < b_-\}, \quad P_1 u_+ \in \{c\varphi_1 : a_+ < c < b_+\}$$

and

$$F(c) < 0 \qquad \text{for } c \in \{c\varphi_1 : a_- < c < b_-\} \cup \{c\varphi_1 : a_+ < c < b_+\}.$$

Here we define subsets $A^{\pm}$ of V by

$$A^{\pm} = \{v \in V : F(v) < 0, < P_1 v, \varphi_1 > \in (a_{\pm}, b_{\pm})\}, \tag{2.3}$$

respectively. We put

$$c_{\pm} = \min\{F(s\varphi_1) : \text{sgn} s = \pm 1\}.$$

For each $i \geq 1$, we denote by $F_i(v)$ the restriction of F to V_i, and by $A(i)_c$ the intersection of level set A_c with V_i. That is

$$A(i)_c = \{v \in V_i : F(v) \leq c\}.$$

We put

$$A_c^{\pm} = \overline{A^{\pm}} \cap A_c \qquad \text{for each } c > 0.$$

Lemma 2. *Let $c < 0$ such that $c_{\pm} < c$. Then*

$A_c^{\pm}$ are nonempty bounded and closed.

Proof. Since $g'(\pm\infty) < \lambda_1$, we have that $F(v) \to \infty$, as $\| v \| \to \infty$. This implies that A_c is bounded. It is obvious from the definition of $A_c^{\pm}$ that $A_c^{\pm}$ are closed. ∎

For each $i \geq 1$, we denote by $A(i)_c^{\pm}$ the restriction of $A_c^{\pm}$ to the subspace V_i. We set

$$K(i)_{\pm} = \overline{co} A(i)_c^{\pm} \qquad \text{and} \qquad K_{\pm} = \overline{co} A_c^{\pm}.$$

Since $A(i)_c^{\pm} \subset A^{\pm}$, we have by (2.3) that

$$K(i)_{+} \cap K(i)_{-} = \phi.$$

Then we have that

Lemma 3. *There exist $c_{\pm}, \bar{c}_{\pm} < 0$ with $c_{\pm} < \bar{c}_{\pm}$ and $d > 0$ such that*

$$\| Lv - g(v) \|_* \geq d \qquad \text{for all } v \in A_{\bar{c}_+}^{+} \backslash A_{c_+}^{+} \cup A_{\bar{c}_-}^{+} \backslash A_{c_-}^{-}. \tag{2.4}$$

Proof. We choose $c_{\pm}$ and $\overline{c}_{\pm}$ such that $cl(A^{\pm}_{\overline{c}_{\pm}} \backslash A^{\pm}_{c_{\pm}})$ are disjoint from the set of critical points of F. It is well known that the functional F satisfies Palais- Smale condition, i.e., any sequence $\{x_n\}$ satisfying $\{F(x_n)\}$ is bounded and $F'(x_n) \to 0$ contains a convergent subsequence. If (2.4) does not hold for any $d > 0$, there exists a sequence $\{x_n\}$ such that

$$x_n \in D = A^+_{\overline{c}} \backslash A^+_c \cup A^-_{\overline{c}} \backslash A^-_c$$

and $F'(x_n) \to 0$, as $n \to \infty$. Since $A^{\pm}_{\overline{c}}$ are bounded, by Palais-Smale condition, we have that there exists a convergence subsequence $\{x_m\}$ of $\{x_n\}$. Let $v \in V$ such that $x_m \to v$. Then we have that $v \in D$ and $\nabla F(v) = 0$. This contradicts the definition of $c_{\pm}$ and $\overline{c}_{\pm}$. ∎

For simplicity of notations, we put $c = c_{\pm}$ and $\overline{c} = \overline{c}_{\pm}$.

Lemma 4. *For each $i \geq 1$, there exist mappings $Q(i)_{\pm} : K(i)_{\pm} \to A(i)^{\pm}_{c}$ such that $Q(i)_{\pm}$ are continuous and*

$$Q(i)_{\pm}x = x \qquad \textit{for each } x \in A(i)^{\pm}_{c}. \tag{2.5}$$

Proof. Fix $i \geq 1$. Let $x \in K(i)_+$. Then x is uniquely decomposed as $x = x_1 + x_2$, where $x_1 \in V_1$ and $x_2 \in V_1^{\perp} \cap V_i$. Then since

$$C_{x_1} = \{v \in V_1^{\perp} \cap V_i : F(x_1 + v) \leq c\}$$

is nonempty and strictly convex by Lemma 2, we have that there exists an unique element $\widetilde{x} \in C_{x_1}$ such that

$$\| x_2 - \widetilde{x} \| = \min\{\| x_2 - y \| : y \in C_{x_1}\}.$$

We put $Q(i)_+x = x_1 + \widetilde{x}$. Then from the definition, it is obvious that $Q(i)_+x \in A(i)^+_c$ and that (2.5) holds. The mapping $Q(i)_-$ is defined by the same way. It is easy to see that $Q(i)_{\pm}$ are continuous on $K(i)_{\pm}$. ∎

3. Proof of Theorem .

We consider initial value problems of the form

$$(I)\qquad \begin{aligned}&\frac{du}{dt} - \Delta u - g(u) = f(t), \quad t > 0\\ &u(0) = u_0, \quad (u_0 \in V),\end{aligned}$$

and

$$(I_i)\qquad \begin{aligned}&\frac{dv}{dt} - \Delta v - P_i g(v) = P_i f(t), \quad t > 0\\ &v(0) = v_0,\end{aligned}$$

where $i \geq 1$ and $v_0 \in V_i$.

We define mappings $T_f : V \to V$ and $T_{f,i} : V_i \to V_i$ by

$$T_f(u_0) = u(T), \qquad \text{and} \qquad T_{f,i}(v_0) = v(T)$$

Then it is easy to verify that T_f and $T_{f,i}$ and continuous on V and V_i. From the definition of T_f, each fixed point u of T_f is a periodic solution of (P). To prove Theorem, we need a few lemmas.

Lemma 5. *There exists a positive number M and such that if* $\sup\{\mid f(t) \mid : t \in [0,T]\} < M$, *then*

$$F_i(v_i(t)) < F_i(v_i)$$

for all $i \geq 1$, $v_i \in D$ and $t > 0$ satisfying

$$v_i(s) \in D \qquad \text{for all } s \in [0,t],$$

where $v_i(\cdot)$ is the solution of (I_i) with $v_0 = v_i$. and $D = A_{\bar{c}}^{+} \backslash A_c^{+} \cup A_{\bar{c}}^{-} \backslash A_c^{-}$.

Proof. *We choose $M > 0$ such that $M < d/2$. Let $i \geq 1$ and v_i be the solution of (I_i) with $v_i(0) = v_i \in D$ and suppose that there exists*

$t > 0$ and $v_i(s) \in D$ for all $s \in [0, t]$. Then by Lemma 4, we have

$$
\begin{aligned}
&F_i(v_i(s)) - F_i(v_i) \\
&\quad = \int_0^s < F'(v_i(\tau)), u_t(\tau) > d\tau \\
&\quad = \int_0^s < Lv_i(\tau) - g(v_i(\tau)), -Lv_i(\tau) + g(v_i(\tau)) + f(\tau) > d\tau \\
&\quad \le \int_0^s (- \mid Lv_i(\tau) - g(v_i(\tau)) \mid^2 + \mid Lv_i(\tau) - g(v_i(\tau)) \mid\mid f(\tau) \mid) d\tau \\
&\quad \le \int_0^s \mid Lv_i(\tau) - g(v_i(\tau)) \mid (- \mid Lv_i(\tau) - g(v_i(\tau)) \mid + \mid f(\tau) \mid) d\tau \\
&\quad \le \int_0^s \| Lv_i(\tau) - g(v_i(\tau)) \|_* (- \| Lv_i(\tau) - g(v_i(\tau)) \|_* + \mid f(\tau) \mid) \\
&\quad \le -(d/2)^2 s + (d/2) \cdot sup\{\mid f(t) \mid : t \in [0, T]\} s < 0
\end{aligned}
$$

∎

From Lemma 5, we have the following lemma.

Lemma 6.

$$T_{f,i}(A(i)_c^{\pm}) \subset int(A(i)_c^{\pm}), \qquad \textit{for each } i \geq 1. \tag{3.1}$$

Proof. Let $i \geq 1$ and $v \in A(i)_c^{+}$. Let v_i be the solution of the problem (I_i) with $v_0 = v$. If there exists an interval $[0, t]$ such that

$$v_i(s) \in D \cap V_i \quad \text{for all } s \in [0, t],$$

then by Lemma 5,

$$F_i(v_i(s)) < F_i(v) \leq c \qquad \text{for all } s \in [0, t]. \tag{3.2}$$

From the definition of $A(i)_c^{+}$, this implies that $v_i(s) \in A(i)_c^{+}$ for all $s \in [0, t]$. Recalling that the boundary $\{v \in V_i : F_i(v) = c\} \cap A(i)_c$ of $A(i)_c$ is contained in D, we obtain from the observation above that

$$F_i(v_i(s)) < F_i(v) \leq c \qquad \text{for all } s > 0.$$

Thus we find that $v_i(s) \in int(A(i)_c^+)$ for all $s > 0$. Then from the definition of $T_{f,i}$, this implies that $T_{f,i}v \in int(A(i)_c^+)$. By the same argument, we have that $T_{f,i}(A(i)_c^-) \subset int(A(i)_c^-)$. ∎

Lemma 7. *For each $i \geq 1$,*

$$deg(I - T_{f,i}, K(i)_\pm, 0) = 1.$$

Proof. Fix $i \geq 1$. We set

$$G_\pm(v) = T_{f,i}Q(i)_\pm v \qquad \text{for } v \in K(i)_\pm.$$

Then by Lemma 6, we have that

$$G_\pm(v) \in int(A(i)_c^\pm) \qquad \text{for all } v \in K(i)_\pm$$

Since $G_\pm$ are continuous mappings on bounded closed convex sets in a finite dimensional space and $G_\pm$ have no fixed point on the boundary of $K(i)_\pm$,

$$deg(I - G_\pm, K(i)_\pm, 0) = 1.$$

From the definition of $G_\pm$ and Lemma 6, we have that the sets of fixed points of $G_\pm$ are contained in $int(A(i)_c^\pm)$, respectively. Then it follows that

$$deg(I - G_\pm, A(i)_c^\pm, 0) = deg(I - G_\pm, K(i)_\pm, 0) = 1.$$

Since $G_\pm = T_{f,i}$ on $A(i)_c^\pm$, we find that

$$deg(I - T_{f,i}, A(i)_c^\pm, 0) = deg(I - G_\pm, A(i)_c^\pm, 0) = 1.$$

This completes the proof. ∎

Lemma 8. *There exists $e > 0$ such that $A_c^+ \cup A_c^- \subset A_e$ and*

$$deg(I - T_{f,i}, A(i)_e, 0) = 1 \qquad \textit{for all } i \geq 1.$$

Proof. Let $e > 0$ such that the set of critical points of F is contained in the interior of A_e. Fix $i \geq 1$. Then since $A_c^+ \cup A_c^- \subset A_e$, we have by Lemma 5 that $T_{f,i}(A(i)_e) \subset int(A(i)_e)$. On the other hand, by the same argument as in Lemma 4, we can define a continuous mapping $Q_e : \overline{co}A(i)_e \to A(i)_e$ such that $Q_e v = v$ for all $v \in A(i)_e$. Then from the same argument as in Lemma 7 with $Q_\pm$ replaced by Q_e, we can see that the assertion follows. ∎

Proof of Theorem. Let $i \geq 1$. Then by Lemma 7, there exist fixed points $v_i^+ \in A(i)_c^+$ and $v_i^+ \in A(i)_c^+$. On the other hand, by Lemma 7 and Lemma 8, we have that

$$deg(I - T_{f,i}, A(i)_e \backslash (A_c^+ \cup A_c^-), 0) = -1.$$

This implies that there exists a fixed point $v_i^0 \in A(i)_e \backslash (A_c^+ \cup A_c^-)$. Now let $\{v_i^\pm\}$ and $\{v_i^0\}$ be sequences obtained by the argument above. Then since $\{v_i^\pm\}$ and $\{v_i^0\}$ are bounded in V, we may assume that $v_i^\pm$ and v_i^0 converge weakly to $v_\pm$ and $v_0 \in V$, respectively. Then it is easy to verify that $v_\pm \in K_\pm$ and $v_0 \in V \backslash (K_+ \cup K_-)$ are fixed points of T_f. This completes the proof. ∎

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